Problem Set 9 due November 18, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: For any angle α, consider the complex number:

$$z = \cos\alpha + i\sin\alpha$$

(1) Compute the product of z with the complex number $z' = \sin\alpha + i\cos\alpha$. Simplify as much as possible! Draw z, z' and zz' on a picture of the complex plane. *(10 points)*

(2) Compute the product of z with the complex number $w = \cos\beta + i\sin\beta$ for any angle β, using the polar form of z and w. Simplify as much as possible! *(5 points)*

(3) Use your result from part (2) to obtain formulas for:

$$\cos(\alpha+\beta) = ...$$
$$\sin(\alpha+\beta) = ...$$

(5 points)

Solution: We can multiply the two complex numbers distributing and using that $i^2 = -1$:

$$\begin{aligned} z \cdot z' &= (\cos\alpha + i\sin\alpha)(\sin\alpha + i\cos\alpha) \\ &= (\cos\alpha\sin\alpha) + i(\sin^2\alpha + \cos^2\alpha) + i^2\sin\alpha\cos\alpha \\ &= (\cos\alpha\sin\alpha - \cos\alpha\sin\alpha) + i(\sin^2\alpha + \cos^2\alpha) \\ &= i \end{aligned}$$

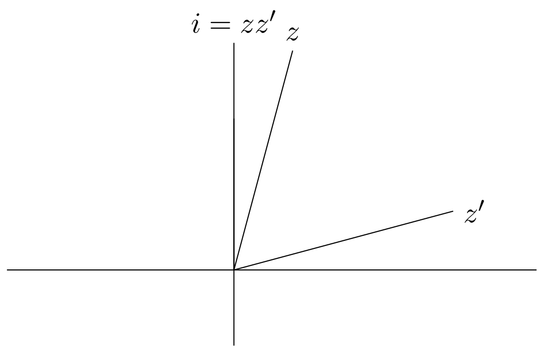
where in the last step we have used the identity $\cos^2\alpha + \sin^2\alpha = 1$. Therefore, if $z' = re^{i\beta}$, we have:

$$zz' = re^{i(\alpha+\beta)} = i$$

hence $r = |i| = 1$ and $\alpha + \beta = \arg i = \frac{\pi}{2}$. We conclude that:

$$z' = e^{i\left(\frac{\pi}{2}-\alpha\right)}$$

In the complex plane, we see that the line bisecting the angle between z and z' is at exactly 45 degrees. We can plot the three points as below:



Grading rubric: 5 points for correct product, 5 points for correct drawing.

Solution: Here we can use the polar form $z = e^{i\alpha}$ and $w = e^{i\beta}$. Then

$$\begin{aligned} zw &= e^{i\alpha}e^{i\beta} \\ &= e^{i(\alpha+\beta)} \\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta). \end{aligned}$$

Grading rubric: 5 points for correct product with partial credit of 2.5 if computational errors.

Solution: Notice that if two complex numbers are equal then both their real and imaginary parts are equal. Then we can write, on one hand

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = e^{i(\alpha+\beta)}$$

while on the other hand,

$$\begin{aligned} e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} &= (\cos\alpha + i\sin\alpha)(\cos(\beta) + i\sin\beta) \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha) \end{aligned}$$

By the "notice" above, we have equality of both the real and imaginary parts:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \end{aligned}$$

Grading rubric: 5 points for correct formulas, 2.5 partial credit for sign/computational errors.

Problem 2: The damped harmonic oscillator (a.k.a. mass on a spring, moving in a straight line in the presence of friction) obeys the following linear differential equation:

$$\underbrace{1}_{\text{mass}} x''(t) + \underbrace{2}_{\text{friction coefficient}} x'(t) + \underbrace{2}_{\text{spring constant}} x(t) = 0$$

(this is math, so no need to assign units to the numbers above). The initial position is $x(0) = 0$ and the initial velocity is $x'(0) = 1$. Find the complete solution $x(t)$ to the second order differential equation above by converting it into a system of two first order differential equations. Write your answer both in terms of complex exponentials <u>and</u> sines and cosines, by converting from one to the other using formula (223) of the lecture notes. *(20 points)*

Solution: We can convert the single second order differential equation to a system of first order equations by defining $y(t) = x'(t)$. The equation is then given by

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}}_{:=A} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The solution is then given by (see (205) in the Lecture notes):

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{At} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{1}$$

Next, we calculate the matrix exponential by diagonalizing. The eigenvalues of A are given by

$$\det(A - \lambda Id) = \det \begin{bmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{bmatrix} = \lambda^2 + 2\lambda + 2$$

so (solving via the quadratic formula)

$$\lambda = 1 \pm i.$$

Then the eigenvectors are given by

$$N \begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} = \mathbb{R} \begin{bmatrix} -1-i \\ 2 \end{bmatrix} \qquad N \begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix} = \mathbb{R} \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$$

and so the diagonalized form (calculating the inverse) is

$$A = \begin{bmatrix} -1+i & -1-i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1-i & 0 \\ 0 & -1+i \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{4} - \frac{i}{4} \\ \frac{i}{2} & \frac{1}{4} - \frac{i}{4} \end{bmatrix}.$$

Hence since $e^{SDS^{-1}} = Se^D S^{-1}$,

$$\begin{aligned} e^{tA} &= \begin{bmatrix} -1+i & -1-i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} e^{t(-1-i)} & 0 \\ 0 & e^{t(-1+i)} \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{4} - \frac{i}{4} \\ \frac{i}{2} & \frac{1}{4} - \frac{i}{4} \end{bmatrix}. \\ &= e^{-t} \begin{bmatrix} -1+i & -1-i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \cos t - i \sin t & 0 \\ 0 & \cos t + i \sin t \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{4} - \frac{i}{4} \\ \frac{i}{2} & \frac{1}{4} - \frac{i}{4} \end{bmatrix}. \\ &= \begin{bmatrix} e^{-t} \cos t + e^{-t} \sin t & e^{-t} \sin t \\ -2e^{-t} \sin t & e^{-t} \cos t - e^{-t} \sin t \end{bmatrix} \end{aligned}$$

Therefore by (1) above, we have in terms of sines and cosines and complex exponentials respectively:

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} e^{-t}\sin t \\ e^{-t}\cos t - e^{-t}\sin t \end{bmatrix} \\ &= \begin{bmatrix} \frac{i}{2}e^{t(-1-i)} - \frac{i}{2}e^{t(-1+i)} \\ \left(\frac{1-i}{2}\right)e^{t(-1-i)} + \left(\frac{1+i}{2}\right)e^{t(-1+i)} \end{bmatrix}. \end{aligned}$$

Notice, of course, that the second entry is the derivative of first, as it should be, since $y = x'$. The solution to the original second order equation is therefore

$$x(t) = e^{-t}\sin t = \frac{i}{2}e^{t(-1-i)} - \frac{i}{2}e^{t(-1+i)}.$$

Grading rubric: 5 points for correct system of linear equations, 5 for correct diagonalization, 5 for correct matrix exponential, 5 for correct answer (2.5 for each of the two forms).

Problem 3: Write the symmetric matrix:

$$S = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{bmatrix}$$

explicitly as $Q\Lambda Q^T$, where Q is orthogonal and Λ is diagonal. Explain all of your steps (*Hint: the characteristic polynomial of a 4×4 matrix is a degree 4 polynomial, and therefore difficult in general to solve; however, in the case at hand, it will be easily possible to find its roots*) *(20 points)*

Solution: We compute the characteristic polynomial of the matrix as

$$\begin{aligned} \det(S - \lambda Id) &= \det \begin{bmatrix} -\lambda & 0 & a & 0 \\ 0 & -\lambda & 0 & b \\ a & 0 & -\lambda & 0 \\ 0 & b & 0 & -\lambda \end{bmatrix} \\ &= -\lambda \det \begin{bmatrix} -\lambda & 0 & b \\ 0 & -\lambda & 0 \\ b & 0 & -\lambda \end{bmatrix} + a \det \begin{bmatrix} 0 & a & 0 \\ -\lambda & 0 & b \\ b & 0 & -\lambda \end{bmatrix} \end{aligned}$$

where we have expanded along the first column. Then, expanding each 3×3 we have

$$\begin{aligned} &= -\lambda\left(-\lambda \det\begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix} + b\det\begin{bmatrix} 0 & -\lambda \\ b & 0 \end{bmatrix}\right) \det + a\left(-a\det\begin{bmatrix} -\lambda & b \\ b & -\lambda \end{bmatrix}\right) \\ &= \lambda^4 - \lambda^2 b^2 - a^2\lambda^2 + a^2 b^2 \\ &= (\lambda^2 - a^2)(\lambda^2 - b^2) \\ &= (\lambda - a)(\lambda + a)(\lambda - b)(\lambda + b). \end{aligned}$$

From which we see the eigenvalues are $\lambda = \pm a, \pm b$.

Now we can read off the eigenvectors:

$$N(S + aId) = N \begin{bmatrix} a & 0 & a & 0 \\ 0 & a & 0 & b \\ a & 0 & a & 0 \\ 0 & b & 0 & a \end{bmatrix} = \mathbb{R} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad N(S - aId) = N \begin{bmatrix} -a & 0 & a & 0 \\ 0 & -a & 0 & b \\ a & 0 & -a & 0 \\ 0 & b & 0 & -a \end{bmatrix} = \mathbb{R} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$N(S + bId) = N \begin{bmatrix} b & 0 & a & 0 \\ 0 & b & 0 & b \\ a & 0 & b & 0 \\ 0 & b & 0 & b \end{bmatrix} = \mathbb{R} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad N(S - bId) = N \begin{bmatrix} -b & 0 & a & 0 \\ 0 & -b & 0 & b \\ a & 0 & -b & 0 \\ 0 & b & 0 & -b \end{bmatrix} = \mathbb{R} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

And the change of basis matrix (from the eigenbasis to the standard basis) is therefore

$$V = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Notice then that V has orthogonal columns, but they are not yet orthonormal as they must be for an orthogonal matrix. We then see we should write

$$V = \sqrt{2}Q := \sqrt{2} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

where Q is now orthogonal. Then the diagonal form is

$$\begin{aligned} S &= VDV^{-1} = (\sqrt{2}Q)D(\frac{1}{\sqrt{2}}Q^{-1}) = QDQ^T \\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & b \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

Grading rubric: 7.5 points for eigenvalues, 7.5 for eigenvectors, 5 for correct diagonalization form.

Problem 4: Let S be a symmetric matrix. Use the fact that $S = Q\Lambda Q^T$, where Q is orthogonal and Λ is the diagonal matrix of eigenvalues, to prove that any diagonal entry of S lies between the smallest and the largest eigenvalue of S. (*Hint: write out the diagonal entries of S explicitly in terms of the entries of Q*) *(15 points)*

Solution:

We can write $S = Q\Lambda Q^T$ for an orthogonal matrix Q. Note the fact that if Q is an orthogonal matrix, so is Q^T. This follows since $(Q^T)^T Q^T = QQ^T = QQ^{-1} = Id$, where we have used the fact that Q is orthogonal to see $Q^T = Q^{-1}$. The consequence of this is that we can write Q^T as a matrix

$$Q^T = \begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \dots & v_n \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

where $v_i = \begin{bmatrix} v_{i1} & v_{i2} & \dots & v_{in} \end{bmatrix}^T$ are an orthonormal set of vectors. Then

$$\begin{aligned} S = Q\Lambda Q^T &= \begin{bmatrix} \dots & v_1 & \dots \\ \dots & v_2 & \dots \\ & \vdots & \\ \dots & v_n & \dots \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \dots & v_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \\ &= \begin{bmatrix} \dots & v_1 & \dots \\ \dots & v_2 & \dots \\ & \vdots & \\ \dots & v_n & \dots \end{bmatrix} \begin{bmatrix} \lambda_1 v_{11} & \lambda_1 v_{21} & & \lambda_1 v_{n1} \\ \lambda_2 v_{12} & \lambda_2 v_{22} & & \lambda_2 v_{n2} \\ \vdots & \vdots & \dots & \vdots \\ \lambda_n v_{1n} & \lambda_2 v_{2n} & & \lambda_n v_{nn} \end{bmatrix}. \end{aligned}$$

In other words, the columns of ΛQ^T are the vectors v_i with λ_1 multiplying the first entry, λ_2 multiplying the second, and so forth. Then we see a diagonal entry $S_i i$ of S is

$$S_{ii} = \sum_j \lambda_j v_{ij}^2 = \lambda_1 v_{i1}^2 + \lambda_2 v_{i2}^2 + \dots + \lambda_n v_{in}^2.$$

Since all the numbers v_{ij}^2 are positive, replacing each λ_i with a smaller one can only decrease the sum and replacing it with a larger one can only increase it. Thus for $\lambda_{small}, \lambda_{big}$ the highest and lowest eigenvalues we have

$$\lambda_{small} = \lambda_{small}\underbrace{(v_{i1}^2 + v_{i2}^2 + \dots + v_{in}^2)}_{=1} \leq \underbrace{\lambda_1 v_{i1}^2 + \lambda_2 v_{i2}^2 + \dots + \lambda_n v_{in}^2}_{=S_{ii}} \leq \lambda_{big}\underbrace{(v_{i1}^2 + v_{i2}^2 + \dots + v_{in}^2)}_{=1} = \lambda_{big}$$

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Grading rubric: 10 points for correctly identifying the diagonal entries of S, 5 points for correct argument of bounds.

Problem 5: Consider the 3×3 symmetric matrix S such that:

$$\begin{bmatrix} x & y & z \end{bmatrix} S \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x - y + 2z)^2$$

for any x, y, z.

(1) Without doing any computations on S, explain why S cannot have full rank. *(5 points)*

(2) Write S out explicitly. *(5 points)*

(3) Compute the eigenvalues and eigenvectors of S. *(10 points)*

(4) Does your answer in part (3) agree with part (1)? Is S positive definite, positive semi-definite, or neither? *(5 points)*

Solution: The quantity $(x-y+2z)^2$ is the energy of the matrix S. Because it is non-negative for all x, y, z, then S is positive semidefinite. But because it can be 0 for x, y, z not all 0 (for example for $x = 1, y = 3, z = 1$), we conclude that S is not positive definite. Hence S has a zero eigenvalue, hence it cannot have full rank.

Solution: When multiplying out the expression

$$\begin{bmatrix} x & y & z \end{bmatrix} S \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

we see the entry S_{11} contributes the coefficient of x^2 to the expression. Likewise S_{22}, S_{33} contribute the y^2, z^2 terms. S_{12} contributes an xy term as does S_{21}. Together, these must form the coefficient of xy in $(x-y+2z)^2$ and since we know they are equal, the must both be half of it. The same applies for the other off-diagonal entries. Thus multiplying out we have

$$(x-y+2z)^2 = x^2 + y^2 + 4z^2 - 2xy + 4xz - 4yz.$$

Therefore

$$S = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}.$$

Grading rubric: 5 points for correct, 2.5 points for correct diagonal entries.

Solution: We can calculate the characteristic polynomial as

$$
\begin{aligned}
\det(S-\lambda Id) &= \det\begin{bmatrix}1-\lambda & -1 & 2\\ -1 & 1-\lambda & -2\\ 2 & -2 & 4-\lambda\end{bmatrix}\\
&= (1-\lambda)\det\begin{bmatrix}1-\lambda & -2\\ -2 & 4-\lambda\end{bmatrix} + 1\det\begin{bmatrix}-1 & -2\\ 2 & 4-\lambda\end{bmatrix} + 2\det\begin{bmatrix}-1 & 1-\lambda\\ 2 & -2\end{bmatrix}\\
&= (1-\lambda)\left[(1-\lambda)(4-\lambda)-4\right] + \left[(\lambda-4)+4\right] + 2\left[2-2(1-\lambda)\right]\\
&= (1-\lambda)(\lambda^2-5\lambda)+\lambda+4\lambda\\
&= -\lambda^3+6\lambda^2\\
&= -\lambda^2(\lambda-6)
\end{aligned}
$$

From which we see the eigenvalues are $\lambda = 6, 0$ with the latter having (algebraic) multiplicity 2. Then the $\lambda = 0, 6$ eigenvectors are given (respectively) by

$$
N\begin{bmatrix}1 & -1 & 2\\ -1 & 1 & -2\\ 2 & -2 & 4\end{bmatrix} = \mathbb{R}\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix} + \mathbb{R}\begin{bmatrix}1\\ 1\\ 0\end{bmatrix} \qquad N\begin{bmatrix}-5 & -1 & 2\\ -1 & -5 & -2\\ 2 & -2 & -2\end{bmatrix} = \mathbb{R}\begin{bmatrix}1\\ -1\\ 2\end{bmatrix}.
$$

Grading rubric: 5 points for eigenvalues, 5 points for eigenvectors.

Solution: S is positive semi-definite, since all the eigenvalues are non-negative. It is not positive definite since some of them are 0. This is in agreement with our conclusions in part 1, since positive semi-definite matrices that are not positive definite must have 0 as an eigenvalue. This is equivalent to having non-empty nullspace, which means the matrix cannot have full rank since it's square.

Grading rubric: 2.5 points for each part.